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NONEXISTENCE OF ALMOST COMPLEX STRUCTURES ON PRODUCTS OF EVEN-DIMENSIONAL SPHERES

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In this paper we prove the following theorem: $S^{2p} \times S^{2q}$ allows an almost complex structure if and only if $(p, q) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (3, 3)$.

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almost complex structure
characteristic classes

As is very well known, Borel and Serre [1] proved in 1953 that S^2 and S^6 are the only even-dimensional spheres allowing almost complex structures. In passing, it may be remarked that Calabi and Eckmann [3] had shown that complex structures do exist on $S^{2p+1} \times S^{2q+1}$. Here we prove the following:

Theorem 1. *The only products of even-dimensional spheres that allow almost complex structures are* $S^2 \times S^2$, $S^2 \times S^6$, $S^6 \times S^2$, $S^6 \times S^6$, $S^2 \times S^4$, $S^4 \times S^2$.

Remark 2. Of course $S^2 \times S^2$, $S^2 \times S^6$, $S^6 \times S^2$ and $S^6 \times S^6$ allow almost complex structures as S^2 and S^6 allow them. It is easy to see that $S^4 \times S^2$ (and therefore $S^2 \times S^4$) is diffeomorphically embeddable in $\mathbb{R}^7$. Therefore by Calabi's result [2], which says that all 6-dimensional orientable manifolds immersed in $\mathbb{R}^7$ allow almost complex structures, we see that $S^4 \times S^2$ and $S^2 \times S^4$ do allow almost complex structures.

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The idea of the proof of the theorem is to use characteristic class theory; the chief tools being the Bott periodicity and the integrality theorem of Bott. The technique was suggested to us by Prof. M.S. Narasimhan and Prof. M.S. Raghunathan. We are extremely grateful for their generous guidance. Helpful conversations with Prof. M.S. Narasimhan and Prof. S. Nag are also gratefully acknowledged. In particular, the last two remarks are due to them.

Definition 3. The *Chern character* $\mathrm{ch}(W)$ of a complex vector bundle W of rank n over a base B is defined to be the formal sum

$$n+\sum_{k=1}^{\infty}\frac{s_k(c_1(W),\ldots,c_n(W))}{k!}\in H^*(B,Q). \tag{1}$$

Here s_k is the polynomial $s_k(c_1,\ldots,c_n)=\sum_{i=1}^{n}t_i^k$, where c_j, the jth Chern class of W, is the jth elementary symmetric function in the variables $t_1,\ldots,t_n$. See Milnor and Stasheff [5, p. 188] for more details.

Proof of Theorem 1. Let $T(S^{2p}\times S^{2q})$ be the tangent bundle of $S^{2p}\times S^{2q}$. Suppose there is an almost complex structure on $S^{2p}\times S^{2q}$. Then we have

$$T(S^{2p}\times S^{2q})\otimes\mathbb{C}=V\oplus\bar{V}, \tag{2}$$

where V is a complex vector bundle and $\bar{V}$ is its conjugate vector bundle; V is isomorphic to $T(S^{2p}\times S^{2q})$ as a real vector bundle.

First of all we claim the following:

Claim. *The Chern character of V is integral, i.e.,*

$$\mathrm{ch}(V)\in H^*(S^{2p}\times S^{2q};\mathbb{Z}).$$

Proof of the Claim. Recall:

(a) The integrality theorem of Bott, which says that for any complex vector bundle W on S^{2k}

$$\mathrm{ch}(W)\in H^*(S^{2k};\mathbb{Z}) \tag{3}$$

(see Husemoller [4, p. 280]).

(b) The Bott periodicity, which says that

$$K(S^{2p})\otimes K(S^{2q})\cong K(S^{2p}\times S^{2q}) \tag{4}$$

(see Husemoller [4, p. 137]).

The multiplicativity of Chern character implies, utilising (a) and (b), that our claim of integrality is true.

Now,

$$H^{2k}(S^{2p}\times S^{2q})=0 \quad \text{for } k\neq 0, p, q, p+q \tag{5}$$

implies $\mathrm{ch}(V)$ is of the form

$$\mathrm{ch}(V)=(p+q)+\frac{s_p}{p!}+\frac{s_q}{q!}+\frac{s_{p+q}}{(p+q)!}. \tag{6}$$

Let u, v be the fundamental cohomology classes of S^{2p} and S^{2q} respectively. Then the total Chern class of V may be written as

$$\mathrm{c}(V) = 1 + au + bv + cuv \tag{7}$$

with $a, b, c \in \mathbb{Z}$. Since the top Chern class yields the Euler characteristic, $c = 4$. We have

$$\mathrm{c}(\bar{V}) = 1 + (-1)^p au + (-1)^q bv + (-1)^{p+q} cuv. \tag{8}$$

Since the tangent bundle of a sphere, and hence of $S^{2p} \times S^{2q}$, is stably trivial, all the Chern classes of $V \oplus \bar{V}$ are zero. So,

$$\mathrm{c}(V) \cdot \mathrm{c}(\bar{V}) = \mathrm{c}(V \oplus \bar{V}) = 1. \tag{9}$$

It follows that if p is even, $a = 0$ (i.e., $c_p = 0$) and if q is even $b = 0$ (i.e., $c_q = 0$). If both p and q are even, we also obtain $c = 0$, a contradiction.

Now suppose p even (and therefore $c_p = 0$) and q odd. Then by Newton's relation (see [5, p. 196])

$$s_n - c_1 s_{n-1} + c_2 s_{n-2} + \cdots + (-1)^n n c_n = 0 \tag{10}$$

we have $s_q = qc_q$ and $s_{p+q} = (p+q)c_{p+q}$ which gives

$$\mathrm{ch}(V) = (p+q) + \frac{b}{(q-1)!} v + \frac{4}{(p+q-1)!} uv. \tag{11}$$

Since this is an integral class, $(p+q-1)!$ divides 4. Hence $p+q \leqslant 3$, so $p = 2$ and $q = 1$.

Similarly, p odd and q even implies $p = 1$ and $q = 2$.

Now suppose p and q both odd. Then (9) yields $ab = c = 4$. We have again by (10) $s_p = pc_p$ and $s_q = qc_q$. Thus

$$\mathrm{ch}(V) = (p+q) + \frac{a}{(p-1)!} u + \frac{b}{(q-1)!} v + xuv$$

for some x. As this is integral, $(p-1)!$ and $(q-1)!$ divide a and b respectively; as $ab = 4$ we deduce $p \leqslant 3$ and $q \leqslant 3$.

This completes the proof. □

Remark 4. The well-known theorem of Borel and Serre [1], which says that "S^2 and S^6 are the only even-dimensional spheres that can allow almost complex structures" follows from our result.

Remark 5. Notice that when using stable triviality of $T(S^{2k})$ we are tacitly assuming that the *standard* differentiable structure of S^{2k} is being used. We do not know how our theorem would be affected in case exotic differentiable structures (if such exist) are imposed on the spheres.

Remark 6. Note that $S^4 \times S^2$ allows almost complex structures and is the trivial S^2 bundle over S^4. In fact, there is a famous twisted S^2 bundle over S^4 which also not only allows almost complex structures but actually allows the complex manifold structure of $\mathbb{CP}^3$. This arises in the Penrose twistor theory. It may be worthwhile to investigate almost complex structures and complex structures on twisted sphere bundles over spheres rather than just in the trivial product bundles treated above.

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